Rainflow Damage for bi-modal power spectra

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Abstract In this paper, analytical formulas are presented for estimating the rainflow fatigue damage of a Gaussian load process with bi-modal spectrum composed of one low-frequency component and a high frequency one. Those formulas are theoretically constructed from the mathematical definition of rainflow and from the statistical properties of Gaussian processes. The theoretical developments are compared on an application example to other simplified formulas for fatigue damage of bimodal spectra. From that comparison, the proposed formulas are shown to give a reasonably conservative estimate with respect to others. Moreover, those formulas are suitable for two slope S-N curve and can be extended to spectra with more than two peaks.

1 INTRODUCTION

As stated in the last ISSC Fatigue and Fracture Committee report (ISSC, 2006), in a number of ship and offshore engineering applications the loading spectrum is the combination of several spectra pertaining to various frequency ranges (*e.g.* combination of a low-frequency natural response of the mooring with the higher frequency wave-induced one). Most of the time, those components are individually narrow-band and the fatigue damage for each of them can be estimated with a reasonable degree of conservatism and at low computational cost. However, the classical methods to compute the fatigue damage of the resulting wide bandwidth spectra, using either narrow-band approximation or the Monte-Carlo simulation, are too conservative or time consuming. Therefore it would be computationally efficient to compute the total fatigue damage from the estimation of the damage of each component.

In this issue, some simplified formulas for bi-modal spectra that are combination of a high frequency component and a low frequency one have been developed. Among them, the Dual-Narrow-Band formula based on equivalent cycle amplitudes (Lotsberg, 2005). Yet, with this formula, an overestimation of the damage can be observed in many occasions, as reported by Huang and Moan (2006). Another formula was derived by Jiao and Moan (1990) for combination of two narrow-band Gaussian process with well separated spectra. But this formula does not deal with spectra that exhibit more than two peaks nor S-N curve with two slopes.

On the basis of the mathematical definition of rainflow (Rychlik, 1987), we propose in this paper two formulas for estimating the rainflow fatigue damage of bi-modal spectra from the damage of each of its components. Those formulas use a splitting of the rainflow ranges into two sets. For one of them the same p.d.f. as for the high frequency component ranges can be used and appropriate theoretical forms that ensure conservatism are proposed for the p.d.f. of rainflow ranges in the second set. The main advantages of this method are that it can deal with many narrow-band low-frequency components by recursively adding their contribution to the total damage, and that since p.d.f's of the ranges are considered, the method can take two-slopes S-N curves into account.

2 FATIGUE DAMAGE ESTIMATES

Let us consider a signal $s(t)$ which is the sum of a high frequency component $h(t)$ with a spectrum $S_h(f)$ of limited but not necessarily narrow bandwidth ($S_h(f) = 0$ for $f < f_1$ and for $f > f_2$) and of a narrow-bandwidth low frequency component $l(t)$ with a spectrum $S_l(f)$ clearly separate from the high frequency one. We introduce the parameters α, ζ, β and η_s as follows:

$$\alpha = \sqrt{\frac{m_{0h}}{m_{0h} + m_{0l}}}, \quad \zeta = \sqrt{\frac{m_{2l} m_{0h}}{m_{0l} m_{2h}}}, \quad \beta = \frac{N_l}{N_s} \quad \text{and} \quad \eta_s = \frac{m_{2s}}{\sqrt{m_{4s} m_{0s}}} \tag{1}$$

where m denotes the spectral moment with a subscript composed of a number for the order of the moment and a letter identifying the corresponding signal, N is the number of local maxima of the corresponding signal. Note that η_s is the irregularity factor of the sum signal.

From the separation assumption, we have $N_s \approx N_h$, which leads, after some algebraic manipulation, to the following expressions for the parameters β and η_s:

$$\beta = \frac{N_l}{N_s} \approx \frac{N_l}{N_h} = \eta_h \zeta \quad \text{and} \quad \eta_s = \frac{m_{2s}}{\sqrt{m_{4s} m_{0s}}} \approx \frac{\alpha^2 + \zeta^2 - \alpha^2 \zeta^2}{\sqrt{\alpha^2 + \left(1-\alpha^2\right) \eta_h{}^2 \zeta^4}} \eta_h \tag{2}$$

We can partition the turning points of signal $s(t)$ into two separate subsets $\mathcal{A}$ and $\mathcal{B}$ (Figure. 1). Let us split the time into the consecutive intervals over which $l(t)$ crosses zero with a positive slope. The turning points belonging to subset $\mathcal{B}$ are the highest maxima and the lowest minima in such intervals. Thus turning points in subset $\mathcal{B}$ will contribute to N_l cycles since $l(t)$ is assumed narrow-banded. The remaining turning points constitute subset $\mathcal{A}$ and will contribute to $N_s - N_l$ cycles. In the case where a local maximum in $\mathcal{B}$ would be negative, it can be replaced with a "positive zero" value with a conservative effect, and similarly a positive local minimum yet in $\mathcal{B}$ shall be replaced with a "negative zero".

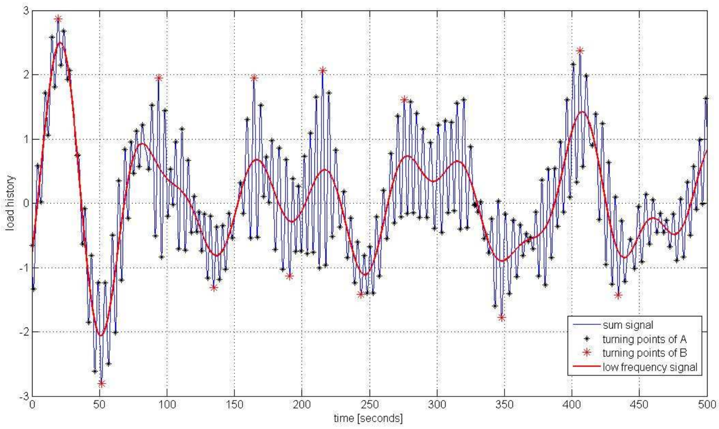

Figure 1. Partition of the set of turning points in two subsets $\mathcal{A}$ anb $\mathcal{B}$.

Using the mathematical definition of rainflow given by Rychlik (1987), it is easy to prove that $\mathcal{B}$ is a "stable" subset in the rainflow count, *i.e.* that any turning point in $\mathcal{B}$ is associated by the counting procedure over the full signal to another turning point also belonging to $\mathcal{B}$. As a consequence, $\mathcal{A}$ is also rainflow-stable. Hence, the sum of the damages associated to each subset can give a reasonably conservative estimate of the total fatigue damage.

2.1 Damage associated to subset $\mathcal{A}$

The rainflow ranges in subset $\mathcal{A}$ can be assumed to have the same distribution as those in the high-frequency signal only. Hence, the associated damage can be estimated as the rainflow damage D_h associated to that high-frequency signal reduced proportionally to the reduction in the number of cycles and taking into account a scaling effect of the high frequency range due to the addition of low frequency signal.

The number of cycles in $\mathcal{A}$ is $N_s - N_l$ as stated above. Thus, the proportional reduction in number is given by $\left(N_s - N_l\right)/N_s = 1 - \beta$. Now, let us denote M_s and M_h the local maxima of respectively the sum signal and its high-frequency component. $\mathrm{E}[M_s] = \mathrm{E}\left[M_h + l\left(t_{M_h}\right)\right] = \mathrm{E}[M_h] + \mathrm{E}\left[l(t) \middle| t = t_{M_h}\right]$. Since low and high frequency component are independent $\mathrm{E}\left[l(t) \middle| t = t_{M_h}\right] = \mathrm{E}[l(t)] = 0$, which leads to $\mathrm{E}[M_s] = \mathrm{E}[M_h]$. Let us denote similarly M_A and M_B the maxima of respectively subset $\mathcal{A}$ and $\mathcal{B}$. The sum of the maxima M_s equals the simple addition of the sum of maxima M_A and M_B. It follows that:

$$\mathrm{E}[M_A] = \left(1 - \frac{\beta}{1-\beta}(\lambda - 1)\right), \quad \lambda = \frac{\mathrm{E}[M_h]}{\mathrm{E}[M_B]} \tag{3}$$

Since the selection process for $\mathcal{B}$ ensures that $\mathrm{E}[M_B] > \mathrm{E}[M_h]$, the amplitude of ranges in $\mathcal{A}$ is reduced with respect to those of high-frequency component. Finally, an estimate of damage associated to subset $\mathcal{A}$ reads:

$$D_A = (1-\beta)\left(1 - \frac{\beta}{1-\beta}(\lambda - 1)\right)^m D_h \tag{4}$$

Note that the following section will give the material to compute $\mathrm{E}[M_B]$ and consequently λ.

2.2 Damage associated to subset $\mathcal{B}$

Subset $\mathcal{B}$ contains N_l cycles. Since the low frequency signal is narrow band, the associated damage reads:

$$D_l = \frac{N_l}{K}\left(2\sqrt{2}\right)^m \sqrt{m_{0l}}^{\,m}\, \Gamma\left(\frac{m}{2}+1\right) \tag{5}$$

Thus

$$D_B = \frac{N_l}{K}\sqrt{m_{os}}^{\,m}\, \mathrm{E}\left[\left(\frac{M_B}{\sqrt{m_{0s}}}\right)^m\right] = \frac{1}{\sqrt{1-\alpha^2}^{\,m}} \frac{\mathrm{E}\left[\left(M_B/\sqrt{m_{0s}}\right)^m\right]}{\left(2\sqrt{2}\right)^m \Gamma(m/2+1)} D_l \tag{6}$$

Two approximations of the distribution of the points in subset $\mathcal{B}$ are considered for the computation of $\mathrm{E}\left[M_B{}^m\right]$. A sensible assumption is that an M_B amplitude is the maximum amongst the local maxima that are found between two successive up-crossings of the signal $l(t)$ and that those local maxima are uncorrelated. The average number of local maxima in such an interval is $N_s/N_l = 1/\beta$. Now, let us remind that the high of local maxima of a Gaussian process with zero mean has a Rice distribution (Rice, 1945), which is the distribution for the sum of a Gaussian and a Rayleigh variable and depends on the irregularity factor of the signal. Let us denote $F_{\max}(x)$ the cumulative distribution of the high of local maxima of the sum signal and $f_{\max}(x)$ the corresponding probability density function. Closed-form expression of those distribution functions can be found in Leadbetter *et al.* (1983, Ch. 15.5). The cumulative distribution $Y_1(x)$ for M_B can be derived from the distribution $F_{\max}(x)$ as follows:

$$Y_1(x) = \left[F_{\max}(x)\right]^{1/\beta}, \quad y_1(x) = \frac{1}{\beta} f_{\max}(x)\left[F_{\max}(x)\right]^{1/\beta - 1} \tag{7}$$

Since Y_1 is an "as exact as possible" approximation for the distribution of the M_B amplitudes, it is likely that it can be used to provide a good estimate of the rainflow damage due to the $\mathcal{B}$ subset, but conservatism is limited to the effect of narrow-band approximation and might be judged insufficient in the case of an effectively very narrow-band M_B history.

The next approximation Y_2 offers a more conservative distribution for the M_B amplitudes and may thus be preferred when a conservative estimate is required. The approximation Y_2 assumes that M_B amplitudes are the largest possible, *i.e.* all belong to the upper β-fractile of the distribution of the maxima of $s(t)$. If we denote $\delta_2 = \max\left(0, F_{\max}{}^{-1}(1-\beta)\right)$, Y_2 reads:

$$Y_2(x) = 1 - \frac{1}{\beta}\left(1 - F_{\max}(x)\right), \quad y_2(x) = \frac{1}{\beta} f_{\max}(x) \quad \text{for} \quad x \ge \delta_2 \tag{8}$$

The moment $\mathrm{E}\left[M_B{}^m\right]$ can be computed by numeric integration using the above approximating distributions. Finally, Denoting $Z_i(m,\alpha,\beta,\eta_s)$ (*e.g.* i = 1,2 depending on which distribution Y_i is used) the normalised moment:

$$Z_i(m,\alpha,\beta,\eta_s) = \frac{\mathrm{E}\left[\left(M_B/\sqrt{m_{0s}}\right)^m\right]}{\left(2\sqrt{2}\right)^m \Gamma(m/2+1)} \tag{9}$$

the total damage can be expressed as:

$$D = D_A + D_B = (1-\beta)\left(1 - \frac{\beta}{1-\beta}(\lambda - 1)\right)^m D_h + \frac{Z_i(m,\alpha,\beta,\eta_s)}{\sqrt{1-\alpha^2}^{\,m}} D_l \tag{10}$$

Note that the distribution Y_1 can be used to estimate the E[M_B] value to compute λ, since this distribution is as close as possible to the distribution of the M_B amplitudes. However, an unconservative estimate of the factor to D_h may enfollow. To unsure always conservative estimate of D_A, a lower bound of λ should be found, then we would advice to use the broadband lower-bound for Y_1, namely with an irregularity factor $\eta_s = 0$.

In case a double slope m_1 and m_2 with intersection point at range S_c is to be considered, the formula becomes an incomplete moment:

$$D_m(S_c) = \frac{N}{K}\int_{S_c}^{\infty} S^m\, p(S)\, dS \tag{11}$$

and damage can be expressed as: $D = D_{m_2}(S_c) + D_{m_1}(0) - D_{m_1}(S_c)$. Care should be taken that the negative term $-D_{m_1}(S_c)$ needs to be unconservatively estimated since it is substracted from the others.

2.3 Reference to other formulas

Some other works have also provided simplified formulas for rainflow fatigue damage of bi-modal spectra from estimation of the damage due to each of the components. In particular we recall here the work by Lostberg (2005) which produced the so-called Dual Narrow-Band formula and the formula derived by Jiao & Moan (1990).

For combination of two general high and low frequency processes, Lostberg (2005), by considering two equivalent damages constant amplitude derived a simple formula which is taken as DnV's rule for evaluating the fatigue damage due to combination of high and low frequency loads.

$$D_{DNB} = (1-\zeta)D_h + \left(1 + \frac{\alpha}{\sqrt{1-\alpha^2}}\right)^m D_l \tag{11}$$

For combination of two narrow-band Gaussian process with well separate spectra, Jiao and Moan (1990) derived a reduced factor by which the narrow-band approximation of the combined fatigue damage is taken into account. In the form using the respective damage of each component, Jiao and Moan formula becomes:

$$\begin{aligned} D_{CNB} = D_h + &\frac{1}{\left(1-\alpha^2\right)^{\frac{m-1}{2}}\zeta}\sqrt{\left(1-\alpha^2\right)\zeta^2 + \alpha^2 {\delta_h}^2} \times \\ &\left[\left(1-\alpha^2\right)^{\frac{m}{2}+1}\left(1-\alpha^2 - \alpha\sqrt{1-\alpha^2}\right) + \alpha\sqrt{\pi\left(1-\alpha^2\right)}\frac{m\Gamma\left(\frac{m+1}{2}\right)}{\Gamma\left(\frac{m}{2}+1\right)}\right] D_l \end{aligned} \tag{12}$$

where dh is the Vanmarcke's bandwidth parameter $\delta_h = \sqrt{1 - \frac{{m_{1h}}^2}{m_{0h}\, m_{2h}}}$ et α^2 is small.

3 PRACTICAL EXAMPLES

3.1 Quasi-static response to swell and wind sea

Let us consider a response spectrum identical to a wave spectrum, composed of a 2.31 meter swell at 20 seconds period that we model by a jonswap spectrum with $\gamma = 10$ and a 2.31 meter wind sea at 5 seconds period that we model by a jonswap spectrum with $\gamma = 3$. The corresponding values of the parameters are: $\alpha = 0.707$, $\zeta = 0.226$, $\beta = 0.159$ and $\eta_s = 0.504$.

Both proposed formulas are applied to this example. The fatigue damage is also computed using other existing formulas, namely the narrow-band approximation, the simple summation, the dual narrow-band formula and the combined narrow-band formula. They are compared to a reference solution which is the rainflow damage over 10 simulations of 3-hours time histories of the composite spectrum. Figure 2 shows the damage predicted by the various formulas normalized by the reference solution and Figure 3 presents the contribution of each subset to the conservatism of our formulas.

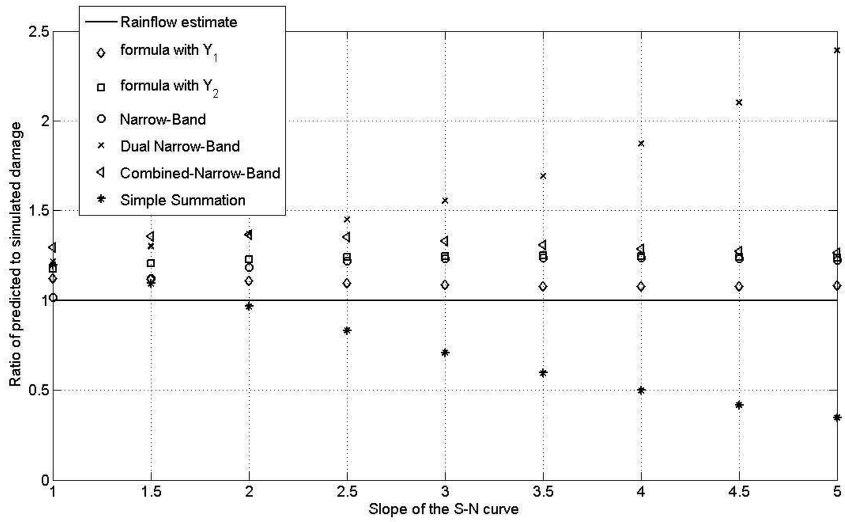

Figure 2. Comparison of formulas for damage estimation

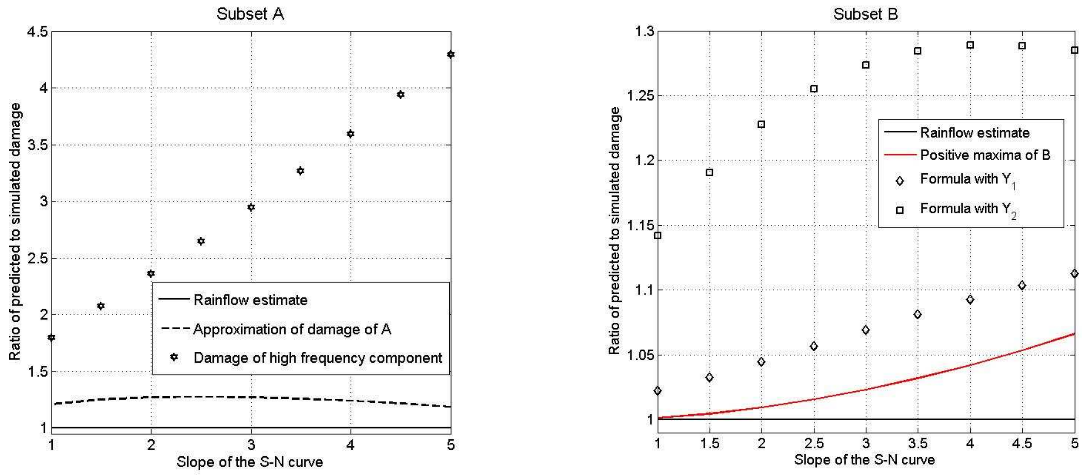

Figure 3. Level of conservatism on damage associated to subset $\mathcal{A}$ and $\mathcal{B}$

We have around 25% of conservatism on the damage due to subset $\mathcal{A}$, the distribution Y_1 yields less than 10% of conservatism on the damage due subset $\mathcal{B}$, while Y_2 gives a conservatism between 15% and 30%. On the total damage the formula with Y_1 is close to the rainflow estimate with less than 10% of conservatism while with Y_2 we have less than 30% of conservatism. Note that both formulas are less conservative than the narrow-band approximation and the Dual Narrow-Band formula, they gives better estimate than the Combined Narrow-Band approximation. Simple summation yields unconservative estimate of the damage.

3.2 Mooring plus Quasi-static response to waves

Let us add to the previous signal a low-frequency mooring response equivalent to a 2.31 meter swell at 70 seconds. We model that signal by a quasi narrow-band triangular spectrum with a bandwidth limited to 2 10^{-3}Hz. The previous signal now represents the high frequency component and it is not especially narrow-banded. The corresponding values of the parameters are: α = 0.816, ζ = 0.076, β = 0.0385 and η_s = 0.408. A recursive addition is applied in this case using the damage value provided in the previous example by the proposed formula as the estimation of the high-frequency damage. Figure 4 shows the damage predicted by the various formulas normalized by the reference solution.

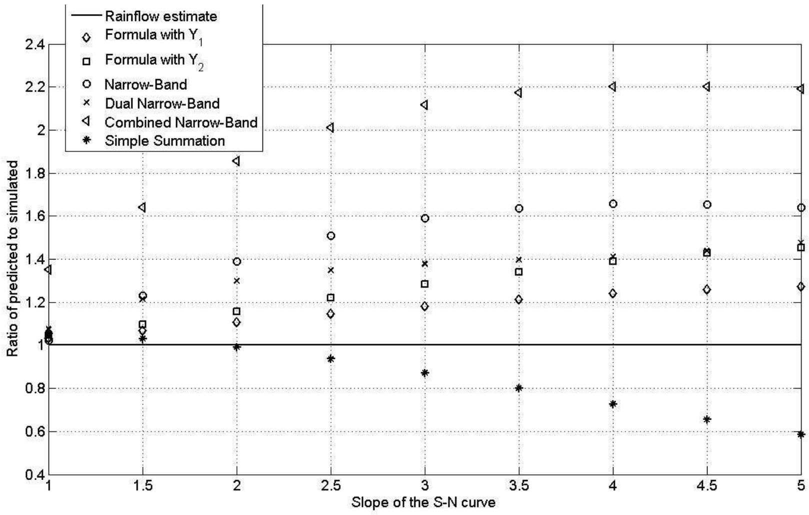

Figure 4. Comparison of formulas for damage estimation

The formula with Y_1 yields less than 25% conservatism while the formula with Y_2 gives a conservatism between 20% and 40%, lower than the Dual Narrow-Band approximation, and still less conservative than the Narrow-Band approximation. The Combined Narrow-Band overestimate in this case the total damage. Simple summation is still unconservative.

4 CONCLUSION

We have given two formulas to compute rainflow damage for composite signals from the individual properties and damages of their high- and low-frequency components. Those formulas are based on the mathematical definition of rainflow and on the theoretical properties of gaussian processes, the first one may be used to provide either best effort estimates, not garanteed to be conservative at all times, or conservative ones, and the second one conservative ones. Those formulas are free from the semi-empiricism that ruled many previous formulas for rainflow damage. They also allow for a wide-band high-frequency component, and thus for iterative addition of several narrow-band low-frequency ones, and for the introduction of two-slopes S-N curves or endurance limits.

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